The best constant of discrete Sobolev inequality

2009 J. Phys. A: Math. Theor. 42 454014

(http://iopscience.iop.org/1751-8121/42/45/454014)

J. Phys. A: Math. Theor. **42** (2009) 454014 (12pp) doi:10.1088/1751-8113/42/45/454014

The best constant of discrete Sobolev inequality

Atsushi Nagai[1], **Yoshinori Kametaka**[2] **and Kohtaro Watanabe**[3]

[1] College of Industrial Technology, Nihon University, 2-11-1 Shin-ei, Narashino 275-8576, Japan
[2] Faculty of Engineering Science, Graduate School of Mathematical Sciences, Osaka University, 1-3 Machikaneyama, Toyonaka 560-8531, Japan
[3] Department of Computer Science, National Defense University, Hashirimizu, Yokosuka 239-8686, Japan

E-mail: nagai.atsushi@nihon-u.ac.jp, kametaka@sigmath.es.osaka-u.ac.jp and wata@nda.ac.jp

Received 27 February 2009, in final form 27 June 2009
Published 27 October 2009
Online at stacks.iop.org/JPhysA/42/454014

Abstract
A discrete version of Sobolev inequalities in Hilbert spaces ℓ^2 and ℓ^2_N, which are equipped with an inner product defined by using $2M$th positive difference operators, is presented. Their best constants are also found by means of the theory of reproducing kernel and are given by a harmonic mean of the spectra of the difference operator. Other expressions of the best constants are also derived.

PACS numbers: 02.30Sa, 02.30Lt
Mathematics Subject Classification: 46E39, 41A44

1. Introduction

The best constant of the Sobolev inequality

$$\|u\|_{L^q(R^N)} \leqslant C\|\nabla u\|_{L^p(R^N)}, \qquad u \in W^{1,p}(R^N),$$

was found by Talenti [6] in the case $1 \leqslant p < N, q = Np/(N-p)$ and by Kametaka *et al* [1, 3] in the case $q = \infty$, $p = 2$. In particular, we obtained the best constant of the Sobolev inequalities which come from a physical background such as the string deflection problem [2] and the ladder electric circuit [4].

In our previous work [5], we derived a discrete analogue of the Sobolev inequality starting from the periodic boundary value problems for the $2M$th-order difference operator. The best constant of the discrete Sobolev inequality is given by means of a discrete analogue of Bernoulli polynomials and Riemann zeta functions. The purpose of this paper is to find a discrete version of Sobolev inequalities and their best constants starting from $2M$th-order difference equations equipped with M complex parameters $z_0, \ldots, z_{M-1}$. We restrict ourselves to a one-dimensional case $N = 1$.

This paper is organized as follows. In section 2, we start with the simplest example corresponding to the case $M = 1$, that is the second-order difference operator. We derive the discrete analogue of the Sobolev inequality and find its best constant by using the theory of reproducing kernels. In section 3, we generalize this result to the case $M \geqslant 2$. Section 4 is devoted to the derivation of the discrete Sobolev inequality corresponding to periodic boundary problems for the same difference operator as in section 3. Finally in section 5, we give concluding remarks.

2. Simple example

Let ℓ^2 be a set of all the infinite sequences $\boldsymbol{u} = {}^t(\ldots, u(i), \ldots)$ $(i \in Z)$ of complex numbers $u(i)$ with finite ℓ^2-norm $\|\boldsymbol{u}\| = \left(\sum_{i=-\infty}^{\infty} |u(i)|^2\right)^{1/2} < \infty$. For any two elements $\boldsymbol{u}$ and $\boldsymbol{v}$ in ℓ^2, we attach a usual unitary inner product:

$$(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=-\infty}^{\infty} u(i)\overline{v(i)}.$$

With respect to this inner product, ℓ^2 is a Hilbert space.

We consider the shift operator L on ℓ^2. L maps an element $\boldsymbol{u} = (\ldots, u(i), \ldots)$ to the shifted element $L\boldsymbol{u} = (\ldots, u(i+1), \ldots)$, whose ith element is $u(i+1)$. If we introduce a vector $\boldsymbol{\delta}_j$ $(j \in Z)$ defined by

$$\boldsymbol{\delta}_j = (\ldots, \delta(i-j), \ldots), \qquad \delta(i) = \begin{cases} 1 & (i = 0), \\ 0 & (\text{else}), \end{cases}$$

the operator L is also given by the convolution by $\boldsymbol{\delta}_{-1}$, that is,

$$L = \boldsymbol{\delta}_{-1} *.$$

Convolution operator $*$ is defined by $(\boldsymbol{u} * \boldsymbol{v})(i) = \sum_{j=-\infty}^{\infty} u(i-j)v(j)$.

Now we introduce a complex number z which satisfies $|z| < 1$. An operator $(I - zL)$ is a generalized difference operator of the first order. It is a bounded linear operator. Its inverse operator is given by the Neumann series,

$$(I - zL)^{-1} = \sum_{k=0}^{\infty} z^k L^k, \tag{1}$$

which is also a bounded linear operator.

Now we introduce a special second-order difference operator given by

$$A = (I - zL)^*(I - zL). \tag{2}$$

The above operator A is bounded linear, self-adjoint and positive definite. A new inner product in ℓ^2,

$$(\boldsymbol{u}, \boldsymbol{v})_A = (A\boldsymbol{u}, \boldsymbol{v}) = ((I - zL)\boldsymbol{u}, (I - zL)\boldsymbol{v}), \tag{3}$$

gives an equivalent metric to the old one $(\cdot, \cdot)$. In fact, we have an inequality

$$(1 - |z|)\|\boldsymbol{u}\| \leqslant \|\boldsymbol{u}\|_A \leqslant (1 + |z|)\|\boldsymbol{u}\| \quad (\boldsymbol{u} \in \ell^2)$$

$$\|\boldsymbol{u}\|_A \equiv \sqrt{(\boldsymbol{u}, \boldsymbol{u})_A} = \sqrt{(A\boldsymbol{u}, \boldsymbol{u})} = \|\boldsymbol{u} - zL\boldsymbol{u}\|.$$

Since $L^* = L^{-1}$ we have

$$A = -\bar{z}L(L^{-1} - \bar{z}^{-1}I)(L^{-1} - zI) = -z_1^{-1}L(L^{-1} - z_0 I)(L^{-1} - z_1 I),$$

where $z_0 = z, z_1 = \overline{z}^{-1}$. Given a characteristic polynomial $p(\lambda) = (\lambda - z_0)(\lambda - z_1)$, we have an expansion formula by partial fractions

$$p(\lambda)^{-1} = \sum_{j=0}^{1} e_j(\lambda - z_j)^{-1}, \qquad e_0 = \frac{1}{p'(z_0)} = -\frac{\overline{z}}{1-|z|^2}, \qquad e_1 = \frac{1}{p'(z_1)} = \frac{\overline{z}}{1-|z|^2}.$$

Using this formula, we have

$$A^{-1} = -z_1 L^{-1} \sum_{j=0}^{1} e_j (L^{-1} - z_j I)^{-1}.$$

It is easy to see that the following equalities hold:

$$(L^{-1} - z_0 I)^{-1} = \sum_{k=0}^{\infty} z_0^k L^{k+1} = \sum_{k=0}^{\infty} z^k L^{k+1}, \tag{4}$$

$$(L^{-1} - z_1 I)^{-1} = -\sum_{k=0}^{\infty} z_1^{-k-1} L^{-k} = -\sum_{k=0}^{\infty} \overline{z}^{k+1} L^{-k}, \tag{5}$$

from which we finally have

$$\begin{aligned} A^{-1} &= -z_1 \left(e_0 \sum_{k=0}^{\infty} z_0^k L^k - e_1 \sum_{k=0}^{\infty} z_1^{-k-1} L^{-k-1} \right) \\ &= \frac{1}{1-|z|^2} \left(\sum_{k=0}^{\infty} z^k L^k + \sum_{k=0}^{\infty} \overline{z}^{k+1} L^{-k-1} \right). \end{aligned}$$

The above relation shows that the inverse operator A^{-1} is a convolution operator by the sequence $(\ldots, A^{-1}(i), \ldots)$ given by

$$\begin{aligned} A^{-1}(i) &= -z_1 \left(e_0 \sum_{k=0}^{\infty} z_0^k \delta(i+k) - e_1 \sum_{k=0}^{\infty} z_1^{-k-1} \delta(i-k-1) \right) \\ &= \frac{1}{1-|z|^2} \left(\sum_{k=0}^{\infty} z^k \delta(i+k) + \sum_{k=0}^{\infty} \overline{z}^{k+1} \delta(i-k-1) \right). \end{aligned} \tag{6}$$

For the set of a Hilbert space ℓ^2 and an inner product $(\boldsymbol{u}, \boldsymbol{v})_A$, the kernel function

$$\begin{aligned} K(i,j) &= A^{-1}(i-j) \\ &= -z_1 \left(e_0 \sum_{k=0}^{\infty} z_0^k \delta(i-j+k) - e_1 \sum_{k=0}^{\infty} z_1^{-k-1} \delta(i-j-k-1) \right) \\ &= \frac{1}{1-|z|^2} \left(\sum_{k=0}^{\infty} z^k \delta(i-j+k) + \sum_{k=0}^{\infty} \overline{z}^{k+1} \delta(i-j-k-1) \right). \end{aligned} \tag{7}$$

is a reproducing kernel. In fact, we can show from (3) and (7) that for any $j \in Z = \{0, \pm 1, \pm 2, \ldots\}$ fixed and for any $\boldsymbol{u} \in \ell^2$, $K(i,j)$, as a function in i, belongs to ℓ^2 and the reproducing relation,

$$(\boldsymbol{u}, K(\cdot, j))_A = u(j) \tag{8}$$

holds. From the theory of reproducing kernel, we have the following theorem.

Theorem 1. *Let z be a complex number which satisfies $|z| < 1$. We can find a positive constant C such that for any $\boldsymbol{u} \in \ell^2$ the discrete Sobolev inequality of the following form holds:*

$$\left(\sup_{j\in\mathbb{Z}} |u(j)|\right)^2 \leqslant C(A\boldsymbol{u}, \boldsymbol{u}). \tag{9}$$

Among such constants C, the best (least) constant is given by

$$C(z) = A^{-1}(0) = \frac{1}{1-|z|^2}. \tag{10}$$

If we replace C by $C(z)$ in the above inequality (9), then the equality holds for $u(i) = K(i, j)$ for any fixed $j \in \mathbb{Z}$.

We call inequality (9) the discrete Sobolev inequality because $(A\boldsymbol{u}, \boldsymbol{u}) = \sum_{n=-\infty}^{\infty} |u(j) - zu(j+1)|^2$ is the square of the difference sequence for $\boldsymbol{u}$.

Proof. [Proof of theorem 1.] Applying the Cauchy–Schwartz inequality to the reproducing relation (8), we have

$$\begin{aligned}\left(\sup_{j\in\mathbb{Z}} |u(j)|\right)^2 &= \left(\sup_{j\in\mathbb{Z}} |(\boldsymbol{u}, K(\cdot, j))_A|\right)^2 \\ &\leqslant \sup_{j\in\mathbb{Z}} (K(\cdot, j), K(\cdot, j))_A \|\boldsymbol{u}\|_A^2 = \sup_{j\in\mathbb{Z}} K(j, j)\|\boldsymbol{u}\|_A^2 = A^{-1}(0)\|\boldsymbol{u}\|_A.\end{aligned}$$

Therefore, the best constant of inequality (9) is given by

$$A^{-1}(0) = \frac{1}{1-|z|^2}$$

where we have put $i = 0$ in (6). The equality in (9) holds for $u(i) = K(i, j)$ with j arbitrarily fixed. □

3. Discrete Sobolev inequality of higher order

We introduce M distinct complex numbers $z_0, \ldots, z_{M-1}$ which satisfy $|z_j| < 1 \quad (0 \leqslant j \leqslant M-1)$ and a bounded linear operator on ℓ^2 defined by

$$A = \prod_{j=0}^{M-1} A(z_j), \tag{11}$$

where $A(z) = (I - zL)^*(I - zL)$ is a bounded linear, self-adjoint, positive-definite operator which was treated in section 1. Operators $A(z_j)$ are mutually commutative, so

$$A = \prod_{j=0}^{M-1} (I - \bar{z}_j L^{-1})(I - z_j L)$$

is also self-adjoint and positive definite. Introducing new parameters $z_M, \ldots, z_{2M-1}$ defined by

$$z_{M+j} = \bar{z}_j^{-1} \qquad (0 \leqslant j \leqslant M-1),$$

we can write

$$A = (-1)^M \left(\prod_{j=0}^{M-1} z_{M+j}^{-1}\right) L^M \prod_{j=0}^{2M-1} (L^{-1} - z_j I).$$

We attach a $2M$th-order characteristic polynomial defined by $p(\lambda) = (\lambda - z_0)(\lambda - z_1)\cdots(\lambda - z_{2M-1})$. According to the theory of Lagrange interpolation polynomials we have the following fundamental expansion by partial fractions:

$$p(\lambda)^{-1} = \sum_{j=0}^{2M-1} e_j(\lambda - z_j)^{-1}, \qquad e_j = 1/p'(z_j) \quad (0 \leqslant j \leqslant 2M-1).$$

Using this formula, we have

$$A^{-1} = (-1)^M \prod_{j=0}^{M-1} z_{M+j} L^{-M} \sum_{j=0}^{2M-1} e_j(L^{-1} - z_j I)^{-1}.$$

From (4) and (5), A^{-1} is rewritten as follows:

$$A^{-1} = (-1)^M \prod_{j=0}^{M-1} z_{M+j} \left[\sum_{j=0}^{M-1} e_j \sum_{k=0}^{\infty} z_j^k L^{k+1-M} - \sum_{j=M}^{2M-1} e_j \sum_{k=0}^{\infty} z_j^{-k-1} L^{-k-M} \right].$$

A^{-1} is a convolution operator by a sequence

$$A^{-1}(i) = (-1)^M \prod_{j=0}^{M-1} z_{M+j} \left[\sum_{j=0}^{M-1} e_j \sum_{k=0}^{\infty} z_j^k \delta(i+k+1-M) \right. \\ \left. - \sum_{j=M}^{2M-1} e_j \sum_{k=0}^{\infty} z_j^{-k-1} \delta(i-k-M) \right].$$

Substitution of $i = 0$ in the above expression gives

$$A^{-1}(0) = (-1)^M \prod_{j=0}^{M-1} z_{M+j} \sum_{j=0}^{M-1} e_j z_j^{M-1}. \tag{12}$$

Owing to the property

$$(e_i) = \left(z_j^i\right)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad (0 \leqslant i, j \leqslant 2M-1),$$

we have

$$A^{-1}(0) = (-1)^M \prod_{j=0}^{M-1} z_{M+j} \left(z_0^{M-1}, \ldots, z_{M-1}^{M-1}, 0, \ldots, 0\right)\left(z_j^i\right)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where $\left(z_0^{M-1}, \ldots, z_{M-1}^{M-1}, 0, \ldots, 0\right)$ and $(0, \ldots, 0, 1)$ are $2M$-dimensional vectors. In general, for any $n \times n$ matrix $A = (a_{ij})$ and $n \times 1$ matrices $\boldsymbol{b} = {}^t(b_0, \ldots, b_{n-1})$ and $\boldsymbol{c} = {}^t(c_0, \ldots, c_{n-1})$ the following formula holds:

$${}^t\boldsymbol{b} A^{-1} \boldsymbol{c} = - \begin{vmatrix} A & \boldsymbol{c} \\ {}^t\boldsymbol{b} & 0 \end{vmatrix} \bigg/ |A|.$$

Applying this formula to (13) we have

$$A^{-1}(0) = (-1)^{M-1} \prod_{j=0}^{M-1} z_{M+j} \left| \begin{array}{ccccccc|c} & & & z_j^i & & & & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \\ \hline z_0^{M-1} & \cdots & z_{M-1}^{M-1} & 0 & \cdots & 0 & & 0 \end{array} \right| \Bigg/ \left| z_j^i \right|,$$

where the numerator is a $(2M+1) \times (2M+1)$ matrix and the denominator is a $2M \times 2M$ matrix. The Laplace expansion of the above numerator with respect to the last column gives

$$A^{-1}(0) = (-1)^{M} \prod_{j=0}^{M-1} z_{M+j} \left| \begin{array}{cccccc} & & z_j^i & & & \\ \hline z_0^{M-1} & \cdots & z_{M-1}^{M-1} & 0 & \cdots & 0 \end{array} \right| \Bigg/ \left| z_j^i \right|. \tag{13}$$

We now arrive at the following conclusion.

Theorem 2. *Let $z_0, \ldots, z_{M-1}$ be M complex numbers which satisfy $|z_j| < 1$ $(0 \leqslant j \leqslant M-1)$. $z_M, \ldots, z_{2M-1}$ are given by $z_{M+j} = \bar{z}_j^{-1}$ $(0 \leqslant j \leqslant M-1)$. We can find a positive constant C such that for any $\boldsymbol{u} \in \ell^2$ the following discrete Sobolev inequality of the Mth order holds:*

$$\left(\sup_{j \in \mathbb{Z}} |u(j)|\right)^2 \leqslant C \left\| \prod_{j=0}^{M-1} (I - z_j L) u \right\|^2. \tag{14}$$

Among such constants C, the best constant is given by

$$C(M; z_0, \ldots, z_{M-1}) = (-1)^{M} \prod_{j=0}^{M-1} z_{M+j} \left| \begin{array}{cccccc} & & z_j^i & & & \\ \hline z_0^{M-1} & \cdots & z_{M-1}^{M-1} & 0 & \cdots & 0 \end{array} \right| \Bigg/ \left| z_j^i \right|. \tag{15}$$

If we replace C by $C(M; z_0, \ldots, z_{M-1})$ in inequality (14), the equality holds for $u(i) = K(i, j) = A^{-1}(i - j)$ with $j \in \mathbb{Z}$ arbitrarily fixed.

As a special case, we have

$$C(1; z_0) = \frac{1}{1 - |z_0|^2}, \qquad C(2; z_0, z_1) = \frac{1 - |z_0 z_1|^2}{|1 - z_0 \bar{z}_1|^2 (1 - |z_0|^2)(1 - |z_1|^2)}.$$

It is interesting to note that from the spectral decomposition

$$A(i) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\sqrt{-1} i y} \prod_{j=0}^{M-1} |1 - z_j \, \mathrm{e}^{-\sqrt{-1} y}|^2 \, \mathrm{d}y,$$

where

$$\lambda(y) = \prod_{j=0}^{M-1} |1 - z_j \, \mathrm{e}^{-\sqrt{-1} y}|^2$$

is the continuous spectrum of the self-adjoint operator A, we also have the spectral decomposition

$$A^{-1}(i) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\sqrt{-1} i y} \prod_{j=0}^{M-1} |1 - z_j \, \mathrm{e}^{-\sqrt{-1} y}|^{-2} \, \mathrm{d}y.$$

Putting $i = 0$ we have

$$A^{-1}(0) = \frac{1}{2\pi}\int_0^{2\pi}\prod_{j=0}^{M-1}|1 - z_j\,\mathrm{e}^{-\sqrt{-1}y}|^{-2}\,\mathrm{d}y. \tag{16}$$

We should remark, however, that it is not easy to perform integration if $M \geqslant 2$.

4. Periodic discrete Sobolev inequality

In this section we add the restriction

$$u(i+N) = u(i) \qquad {}^{\forall} i \in \mathbb{Z} \tag{17}$$

for any fixed $N = 2, 3, 4, \ldots$. We modify the definition of a Hilbert space ℓ^2, an inner product $(\boldsymbol{u}, \boldsymbol{v})$ and delta function vector $\boldsymbol{\delta}$ in the previous section as follows:

$$\ell^2_N = \{\boldsymbol{u} = {}^t(u(0), \ldots, u(N-1)) | u(i) \in \mathbb{C} \quad (0 \leqslant i \leqslant N-1)\} = \mathbb{C}^N,$$

$$(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=0}^{N-1} u(i)\overline{v(i)},$$

$$\boldsymbol{\delta}_j = (\ldots, \delta(i-j), \ldots), \qquad \delta(i) = 1 \ (i \equiv 0 \bmod N), \quad 0 \ (\text{else}).$$

The unitary shift operator $L = \boldsymbol{\delta}_{-1}*$ is the cyclic rotate left operator defined by

$$L\,{}^t(u(0), \ldots, u(N-1)) = {}^t(u(1), \ldots, u(N-1), u(0)).$$

The convolution $\boldsymbol{u} * \boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v} \in \mathbb{C}^N$ is obtained by

$$(\boldsymbol{u} * \boldsymbol{v})(i) = \sum_{j=0}^{N-1} u(i-j)v(j)$$

extending the definition of $u(i) \in \mathbb{C}^N$ outside $0 \leqslant i \leqslant N-1$ by periodicity $u(i+N) = u(i) (i \in \mathbb{Z})$. Note that we must add the new restriction

$$L^N = I.$$

The formula of the inverse operator (1) is replaced by

$$(I - zL)^{-1} = (1 - z^N)^{-1}\sum_{k=0}^{N-1} z^k L^k. \tag{18}$$

We note that we have dealt with the same problem putting $z = 1$ in our previous work [5], where we considered a generalized inverse of $I - L$.

We also treat the same form of the operator A (11) as in the previous section but the formula of its inverse is replaced by

$$A^{-1} = (-1)^M \prod_{j=0}^{M-1} z_{M+j}\left[\sum_{j=0}^{M-1} e_j\left(1 - z_j^N\right)^{-1}\sum_{k=0}^{N-1} z_j^k L^{k-M_0} - \sum_{j=M}^{2M-1} e_j\left(1 - z_j^{-N}\right)^{-1}\sum_{k=0}^{N-1} z_j^{-k-1} L^{-k-1-M_0}\right], \tag{19}$$

where $M_0 = \mathrm{Mod}(M-1, N)$ is the remainder of the natural number $M-1$ divided by N and takes values in $\{0, 1, 2, \ldots, N-1\}$. A^{-1} is also a convolution operator:

$$(A^{-1}\boldsymbol{u})(i) = \sum_{j=0}^{N-1} A^{-1}(i,j)u(j) = \sum_{j=0}^{N-1} A^{-1}(i-j)u(j).$$

It should be noted that the relation $A^{-1}(i+N) = A^{-1}(i)$ holds for any $i \in \mathbb{Z}$. Now we have

$$A^{-1}(i) = (-1)^M \prod_{j=0}^{M-1} z_{M+j} \left[\sum_{j=0}^{M-1} e_j \left(1 - z_j^N\right)^{-1} \sum_{k=0}^{N-1} z_j^k \delta(i+k-M_0) \right.$$
$$\left. - \sum_{j=M}^{2M-1} e_j \left(1 - z_j^{-N}\right)^{-1} \sum_{k=0}^{N-1} z_j^{-k-1} \delta(i-k-1-M_0) \right]. \tag{20}$$

Putting $i = 0$, we have

$$A^{-1}(0) = (-1)^M \prod_{j=0}^{M-1} z_{M+j} \left[\sum_{j=0}^{M-1} e_j \left(1 - z_j^N\right)^{-1} \sum_{k=0}^{N-1} z_j^k \delta(k-M_0) \right.$$
$$\left. - \sum_{j=M}^{2M-1} e_j \left(1 - z_j^{-N}\right)^{-1} \sum_{k=0}^{N-1} z_j^{-k-1} \delta(-k-1-M_0) \right]$$
$$= (-1)^M \prod_{j=0}^{M-1} z_{M+j} \sum_{j=0}^{2M-1} e_j \left(1 - z_j^N\right)^{-1} z_j^{M_0}. \tag{21}$$

Through the same argument as in the previous section, $A^{-1}(0)$ is rewritten in the following determinant expression:

$$A^{-1}(0) = (-1)^M \prod_{j=0}^{M-1} z_{M+j} \left| \begin{array}{c} z_j^i \\ \hline \varphi(z_j) \end{array} \right| \Big/ \left| z_j^i \right|, \tag{22}$$

where

$$\varphi(z) = \frac{z^{M_0}}{1 - z^N}.$$

We finally have the following theorem.

Theorem 3. *Let N, M be two fixed numbers $N = 2, 3, 4, \ldots$ and $M = 1, 2, 3, \ldots$. We choose M distinct complex numbers $z_0, \ldots, z_{M-1}$ such that $|z_j| < 1$ $(0 \leqslant j \leqslant M-1)$ and define z_{M+j} by $z_{M+j} = \bar{z}_j^{-1}$ $(0 \leqslant j \leqslant M-1)$. For any vector $\boldsymbol{u} \in \mathbb{C}^N$ there exists a positive constant C such that the following discrete periodic Sobolev inequality holds:*

$$\left(\max_{0 \leqslant j \leqslant N-1} |u(j)| \right)^2 \leqslant C \left\| \prod_{j=0}^{M-1} (I - z_j L) \boldsymbol{u} \right\|^2. \tag{23}$$

Among such constants C the best constant is given by

$$C(N, M; z_0, \ldots, z_{M-1}) = (-1)^M \prod_{j=0}^{M-1} z_{M+j} \left| \begin{array}{c} z_j^i \\ \hline \varphi(z_j) \end{array} \right| \Big/ \left| z_j^i \right|, \tag{24}$$

where $\varphi(z) = (1 - z^N)^{-1} z^{\mathrm{Mod}(M-1,N)}$. If we replace C by $C(N, M; z_0, \ldots, z_{M-1})$ in the above inequality (23), the equality holds for $u(i) = K(i, j) = A^{-1}(i-j)$ for any fixed j $(0 \leqslant j \leqslant N-1)$.

Explicit forms of the best constants in the case $M = 1, 2$ are given in the following theorem, which is proved in the appendix.

Theorem 4. *For any fixed* $N = 2, 3, 4, \ldots$, *the best constant* $C(N, 1; z_0)$ *and* $C(N, 2; z_0, z_1)$ *are given by*

$$C(N, 1; z_0) = \frac{1 - |z_0|^{2N}}{(1 - |z_0|^2)\left|1 - z_0^N\right|^2} > 0, \tag{25}$$

$$C(N, 2; z_0, z_1) = \frac{\sum_{k=0}^{N-1} \left| \sum_{\substack{i+j=k \\ i,j\geqslant 0}} z_0^i z_1^j + \sum_{\substack{i+j=k+N \\ i,j>k}} z_0^i z_1^j \right|^2}{\left|1 - z_0^N\right|^2 \left|1 - z_1^N\right|^2} > 0. \tag{26}$$

It is interesting to note that $A^{-1}(0)$ given by (22) has another expression, which follows from the spectral decomposition or decomposition into Jordan canonical form. Using

$$\omega = \exp(\sqrt{-1}\, 2\pi/N), \qquad W = \frac{1}{\sqrt{N}}(\omega^{ij}), \qquad \hat{L} = (\omega^i \delta_{ij}),$$

we have the decomposition

$$L = W \hat{L} W^*.$$

The diagonal matrix $\hat{L}$ is the Jordan canonical form of L; W is a unitary matrix, so we have $W^* = W^{-1}$. Using this fact, we have the decomposition

$$A = W \hat{A} W^*,$$

where

$$\hat{A} = \prod_{k=0}^{M-1} (I - z_k \hat{L})^* (I - z_k \hat{L}) = \left(\prod_{k=0}^{M-1} |1 - z_k \omega^i|^2 \delta_{ij} \right)$$

is the Jordan canonical form of A. $\lambda_i = \prod_{k=0}^{M-1} |1 - z_k \omega^i|^2$ $(0 \leqslant i \leqslant N-1)$ are eigenvalues of A. From this fact, we have

$$A^{-1} = W \hat{A}^{-1} W^*, \qquad \hat{A}^{-1} = \left(\prod_{k=0}^{M-1} |1 - z_k \omega^i|^{-2} \delta_{ij} \right),$$

that is,

$$A^{-1}(i - j) = \frac{1}{N} \sum_{l=0}^{N-1} \omega^{(i-j)l} \prod_{k=0}^{M-1} |1 - z_k \omega^l|^{-2}.$$

Finally, we have

$$A^{-1}(0) = \frac{1}{N} \sum_{l=0}^{N-1} \prod_{k=0}^{M-1} |1 - z_k \omega^l|^{-2}.$$

5. Concluding remarks

We have derived the discrete Sobolev inequality which estimates the supremum of the sequence $u(j)$ by the norm of difference sequence $\|\boldsymbol{u}\|_A$, which is considered as a discrete analogue of the Sobolev norm. It is expected that the obtained results have important relations with error analysis in the field of numerical analysis.

In the continuous limit, we essentially obtain the boundary value problem for $2M$th differential operator $\prod_{j=0}^{M-1}(-(\mathrm{d}/\mathrm{d}x)^2 + q_j)$, where q_j is constant. In the case $q_j = 0$ $(0 \leqslant j \leqslant M-1)$, we have already obtained the result [3]. In the case $M = 1$ and $q_0 \neq 0$, we have a string deflection problem [2]. It is also an interesting problem to investigate in detail the case $M = 2$, $q_0, q_1 \neq 0$, which describes a beam deflection problem.

Acknowledgments

One of the authors (AN) is supported by J. S. P. S. Grant-in-Aid for Scientific Research (C) No 20540138.

Appendix.

Proof of theorem 4. We here prove theorem 4 concerning the best constant of discrete Sobolev inequality with periodic boundary condition. Since (25) is proved through simple calculations, we prove only (26):

$$C(N,2;z_0,z_1) = z_2 z_3 \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ z_0 & z_1 & z_2 & z_3 \\ z_0^2 & z_1^2 & z_2^2 & z_3^2 \\ \frac{z_0}{1-z_0^N} & \frac{z_1}{1-z_1^N} & \frac{z_2}{1-z_2^N} & \frac{z_3}{1-z_3^N} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ z_0 & z_1 & z_2 & z_3 \\ z_0^2 & z_1^2 & z_2^2 & z_3^2 \\ z_0^3 & z_1^3 & z_2^3 & z_3^3 \end{vmatrix}}. \tag{A.1}$$

Noting that the denominator is calculated as

$$\prod_{i<j}(z_i - z_j) = -\frac{|z_0 - z_1|^2|1 - z_0\bar{z}_1|^2(1-|z_0|^2)(1-|z_1|^2)}{\bar{z}_0{}^3\bar{z}_1{}^3},$$

we have

$$C(N,2;z_0,z_1) = -\frac{|z_0|^2|z_1|^2 \begin{vmatrix} 1/z_0 & 1/z_1 & \bar{z}_0 & \bar{z}_1 \\ 1 & 1 & 1 & 1 \\ z_0 & z_1 & 1/\bar{z}_0 & 1/\bar{z}_1 \\ \frac{1}{1-z_0^N} & \frac{1}{1-z_1^N} & \frac{1}{1-\bar{z}_0^{-N}} & \frac{1}{1-\bar{z}_1^{-N}} \end{vmatrix}}{|z_0 - z_1|^2|1 - z_0\bar{z}_1|^2(1-|z_0|^2)(1-|z_1|^2)}. \tag{A.2}$$

Expanding the numerator with respect to the last row, we have

$$-|z_0|^2|z_1|^2 \begin{vmatrix} 1/z_0 & 1/z_1 & \bar{z}_0 & \bar{z}_1 \\ 1 & 1 & 1 & 1 \\ z_0 & z_1 & 1/\bar{z}_0 & 1/\bar{z}_1 \\ \frac{1}{1-z_0^N} & \frac{1}{1-z_1^N} & \frac{1}{1-\bar{z}_0^{-N}} & \frac{1}{1-\bar{z}_1^{-N}} \end{vmatrix} = -\begin{vmatrix} 1 & 1 & \bar{z}_0^2 & \bar{z}_1^2 \\ z_0 & z_1 & \bar{z}_0 & \bar{z}_1 \\ z_0^2 & z_1^2 & 1 & 1 \\ \frac{z_0}{1-z_0^N} & \frac{z_1}{1-z_1^N} & \frac{-\bar{z}_0^{N+1}}{1-\bar{z}_0^N} & \frac{-\bar{z}_1^{N+1}}{1-\bar{z}_1^N} \end{vmatrix}$$

$$
\begin{aligned}
&= \frac{z_0}{1-z_0^N}\left(\bar z_0 + \bar z_0^2 z_1|z_1|^2 + |z_1|^2\bar z_1 - \bar z_0|z_1|^4 - \bar z_0^2 z_1 - \bar z_1\right)\\
&\quad - \frac{z_1}{1-z_1^N}\left(\bar z_0 + |z_0|^4\bar z_1 + z_0\bar z_1^2 - |z_0|^2 z_0\bar z_1^2 - |z_0|^2\bar z_0 - \bar z_1\right)\\
&\quad - \frac{\bar z_0^{N+1}}{1-\bar z_0^N}\left(z_1 + z_0|z_1|^4 + z_0^2\bar z_1 - z_0^2 z_1|z_1|^2 - |z_1|^2 z_1 - z_0\right)\\
&\quad + \frac{\bar z_1^{N+1}}{1-\bar z_1^N}\left(z_1 + |z_0|^2 z_0 + |z_0|^2\bar z_0 z_1 - |z_0|^4 z_1 - \bar z_0 z_1^2 - z_0\right)\\
&= \frac{1}{\left|1-z_0^N\right|^2|1-z_1^N|^2}\Big\{|z_0|^2(1-|z_0|^{2N})(1-|z_1|^2)|1-z_0\bar z_1|^2\left|1-z_1^N\right|^2\\
&\quad + |z_1|^2(1-|z_1|^{2N})(1-|z_0|^2)|1-z_0\bar z_1|^2\left|1-z_0^N\right|^2\\
&\quad - z_0\bar z_1(1-(z_0\bar z_1)^N)(1-|z_0|^2)(1-|z_1|^2)(1-\bar z_0 z_1)\left(1-\bar z_0^N\right)\left(1-z_1^N\right)\\
&\quad - \bar z_0 z_1(1-(\bar z_0 z_1)^N)(1-|z_0|^2)(1-|z_1|^2)(1-z_0\bar z_1)\left(1-z_0^N\right)\left(1-\bar z_1^N\right)\Big\}
\end{aligned}
$$

through straightforward calculations. Substitution of the above result into (A.2) gives

$$
\begin{aligned}
C(N,2;z_0,z_1) &= \frac{1}{|z_0-z_1|^2(1-|z_0|^2)(1-|z_1|^2)|1-z_0\bar z_1|^2\left|1-z_0^N\right|^2\left|1-z_1^N\right|^2}\\
&\quad\times\Big\{|z_0|^2(1-|z_0|^{2N})(1-|z_1|^2)|1-z_0\bar z_1|^2\left|1-z_1^N\right|^2\\
&\quad + |z_1|^2(1-|z_1|^{2N})(1-|z_0|^2)|1-z_0\bar z_1|^2\left|1-z_0^N\right|^2\\
&\quad - z_0\bar z_1(1-(z_0\bar z_1)^N)(1-|z_0|^2)(1-|z_1|^2)(1-\bar z_0 z_1)\left(1-\bar z_0^N\right)\left(1-z_1^N\right)\\
&\quad - \bar z_0 z_1(1-(\bar z_0 z_1)^N)(1-|z_0|^2)(1-|z_1|^2)(1-z_0\bar z_1)\left(1-z_0^N\right)\left(1-\bar z_1^N\right)\Big\}\\
&= \frac{1}{|z_0-z_1|^2\left|1-z_0^N\right|^2\left|1-z_1^N\right|^2}\\
&\quad\times\left(\frac{|z_0|^2(1-|z_0|^{2N})}{1-|z_0|^2}\left|1-z_1^N\right|^2 + \frac{|z_1|^2(1-|z_1|^{2N})}{1-|z_1|^2}\left|1-z_0^N\right|^2\right.\\
&\quad\left. - z_0\bar z_1\frac{1-(z_0\bar z_1)^N}{1-z_0\bar z_1}\left(1-z_1^N\right)\left(1-\bar z_0^N\right) - \bar z_0 z_1\frac{1-(\bar z_0 z_1)^N}{1-\bar z_0 z_1}\left(1-z_0^N\right)\left(1-\bar z_1^N\right)\right)\\
&= \frac{1}{|z_0-z_1|^2\left|1-z_0^N\right|^2\left|1-z_1^N\right|^2}\sum_{k=0}^{N-1}\left(|z_0|^{2(k+1)}\left|1-z_1^N\right|^2 + |z_1|^{2(k+1)}\left|1-z_0^N\right|^2\right.\\
&\quad\left. - (z_0\bar z_1)^{k+1}\left(1-z_1^N\right)\left(1-\bar z_0^N\right) - (\bar z_0 z_1)^{k+1}\left(1-z_0^N\right)\left(1-\bar z_1^N\right)\right)\\
&= \frac{1}{|z_0-z_1|^2\left|1-z_0^N\right|^2\left|1-z_1^N\right|^2}\sum_{k=0}^{N-1}\left|z_0^{k+1}\left(1-z_1^N\right) - z_1^{k+1}\left(1-z_0^N\right)\right|^2\\
&= \frac{1}{\left|1-z_0^N\right|^2\left|1-z_1^N\right|^2}\sum_{k=0}^{N-1}\left|\sum_{\substack{i+j=k\\ i,j\geqslant 0}} z_0^i z_1^j + \sum_{\substack{i+j=k+N\\ i,j>k}} z_0^i z_1^j\right|^2,
\end{aligned}
$$

which completes the proof.

□

References

[1] Kametaka Y, Watanabe K and Nagai A 2005 *Proc. Japan. Acad.* A **81** 1

[2] Kametaka Y, Watanabe K, Nagai A, Yamagishi H and Takemura K 2007 *Sci. Math. Japonicae Online* **e-2007** 283

[3] Kametaka Y, Yamagishi H, Watanabe K, Nagai A and Takemura K 2007 *Sci. Math. Japonicae Online* **e-2007** 63

[4] Kametaka Y, Takemura K, Yamagishi H, Nagai A and Watanabe K 2007 *Sci. Math. Japonicae Online* **e-2007** 739

[5] Nagai A, Kametaka Y, Yamagishi H, Watanabe K and Takemura K 2008 *Funkcialaj Ekvacioj* **51** 307

[6] Talenti G 1976 *Ann. Mat. Pura. Appl.* **110** 353